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On Segré Products and Applications

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INTRODUCTION

Chow [3] considers the question, if R and S are Noetherian graded Cohen–Macaulay K-algebras, under what circumstances is their Segré product also Cohen–Macaulay. We now wish to pursue these considerations from the following two geometrical perspectives:

Let V and W be arithmerically Cohen–Macaulay varieties in the projective space $\mathbb{P}_K{}^n$ over an infinite field K. We state additional necessary and sufficient conditions when the Segré embedding of $V \times W$ is again an arithmetically Cohen–Macaulay variety. In addition, we investigate when the Segré embedding $S(V \times W)$ is an arithmetically Buchsbaum variety. We obtain a very useful criterion for complete intersections (see Corollary 1).

Generally, the theorem of this paper yields the criterion for Noetherian graded Cohen–Macaulay modules over a graded K-algebra. The statement (i) of this theorem yields Chow's main theorem for our graded K-algebras (see Corollary 2). We also get the statement (i) of [22, Theorem 4.4.4]. Our theorem implies the following Corollary.

COROLLARY. *Let $V \subseteq \mathbb{P}^n$ and $W \subseteq \mathbb{P}^m$ be arithmetically Cohen–Macaulay varieties with dimension $\geqslant 1$. Then we get for the Segré embedding $S(V \times W)$:*

(i) *$S(V \times W)$ is an arithmetically Cohen–Macaulay variety if and only if the arithmetic genus (see [23]) of V and W is zero.*

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(ii) *$S(V \times W)$ is an arithmetically Buchsbaum variety if and only if the regularity index (see Section* 1) *of V and W is* $\leqslant 1$.

Since for nonsingular curves the arithmetic genus agrees with the usual genus, we obtain by the application of this Corollary better information than Seidenberg [15] for the construction of arithmetically normal irregular surfaces free of singularities (see Proposition 8 and Example 2). Our observations even yield for every given dimension arithmetically normal nonsingular projective varieties, which are nonarithmetically Cohen–Macaulay, but arithmetically Buchsbaum varieties (see Proposition 9).

Furthermore, we note that the Segré embedding $S(V \times W)$ is a locally Cohen–Macaulay variety iff V and W are locally Cohen–Macaulay varieties (see Proposition 6). Of course, this statement is true because the local behavior of $V \times W$ does not depend on the embedding, so the theorem that the tensor product of two K-algebras is Cohen–Macaulay iff the factors can be applied. Using our Künneth relation we will give a direct proof. An example shows that in general $S(V \times W)$ is not a locally Buchsbaum variety, when V and W are locally Buchsbaum varieties (see Proposition 7).

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1. Notation and Preliminary Results

We assume throughout that K is an infinite field. By a graded K-algebra we understand a Noetherian graded ring $R = \bigoplus_{i\geqslant 0} R_i$ satisfying the following conditions:

(i) $R_0 = K$ (i.e., every R_i is a vector space over K);

(ii) the maximal ideal $\mathfrak{m} := \bigoplus_{i\geqslant 1} R_i$ is generated by R_1.

With the usual notation, let $M := \bigoplus_{i\in\mathbb{Z}} M_i$ be a graded R-module. Thereby we let $[M]_i$ denote the ith graded part of M, i.e., $[M]_i = M_i$. Let p be an integer and M a graded R-module. We let $M(p)$ denote the graded R-module whose underlying module is the same as that of M and whose grading is given by

$$[M(p)]_i = [M]_{p+i} \qquad \text{for all integers } i.$$

It is well known that the vector spaces $[M]_i$ have finite dimension for all integers i whenever M is a Noetherian or Artinian graded R-module. If M is a Noetherian graded R-module, we let $H(n, M)$ for all $n \in \mathbb{Z}$ denote the Hilbert function of M, i.e.,

$$H(n, M) = \operatorname{rank}_K([M]_n).$$

Since M is Noetherian there is a polynomial $h(t, M) \in \mathbb{Q}[t]$ and an integer n_0

such that $H(n, M) = h(n, M)$ for all $n \geqslant n_0$. The smallest such integer is called the regularity index of M, which we denote by $r(M)$.

For an arbitrary graded R-module M we put

$$a(M) := \inf_{n \in \mathbb{Z}} \{n \mid [M]_n \neq 0\}$$

and

$$e(M) := \sup_{n \in \mathbb{Z}} \{n \mid [M]_n \neq 0\}.$$

If $M \neq 0$ is Noetherian or Artinian then $a(M)$ resp. $e(M)$ is finite.

Let M, N be two graded R-modules. By $\operatorname{Hom}_R(N, M)$ we denote the set of homomorphisms f of the underlying modules such that $f([N]_i) \subseteq [M]_i$ for all $i \in \mathbb{Z}$. The restriction of f to $[N]_i$ we denote by $[f]_i$. The derived functors of $\operatorname{Hom}_R(N, M)$ are $\operatorname{Ext}_R^i(N, M)$. $\underline{\operatorname{Ext}}_R{}^i(N, M)$ denotes the graded R-module given by

$$[\underline{\operatorname{Ext}}_R{}^i(N, M)]_n = \operatorname{Ext}_R{}^i(N, M(n)) \qquad \text{for all } n \in \mathbb{Z} \quad \text{and} \quad i \geqslant 0.$$

For all $i \geqslant 0$ let $H^i(R, M)$ or, if confusions are excluded, $H^i(M)$, denote the K-vector-space

$$\varinjlim_n \operatorname{Ext}_R{}^i \Big(\bigoplus_{j \geqslant n} R_j \, , M \Big) = \varinjlim_n \operatorname{Ext}_R{}^i(\mathfrak{m}^n, M)$$

(the corresponding maps are given by the inclusions $\mathfrak{m}^{n+1} \subseteq \mathfrak{m}^n$). Now let $\underline{H}^i(R, M)$, or briefly $\underline{H}^i(M)$, be defined by $\bigoplus_{p \in \mathbb{Z}} H^i(R, M(p))$, i.e.,

$$\underline{H}^i(M) = \varinjlim_n \underline{\operatorname{Ext}}_R{}^i(\mathfrak{m}^n, M).$$

$\underline{H}^i(M)$ is a graded R-module with $[\underline{H}^i(M)]_n = H^i(M(n))$. As usual we define the local cohomology $\underline{H}_{\mathfrak{m}}{}^i(M)$ of M with support in the maximal ideal $\mathfrak{m}$ of R by

$$H_{\mathfrak{m}}{}^i(M) := \varinjlim_n \underline{\operatorname{Ext}}_R{}^i(R/\mathfrak{m}^n, M).$$

Thus, for $i \geqslant 1$ we have isomorphisms

$$\underline{H}^i(M) \simeq \underline{H}_{\mathfrak{m}}^{i+1}(M)$$

and an exact sequence

$$0 \to \underline{H}_{\mathfrak{m}}{}^0(M) \to M \to \underline{H}^0(M) \to \underline{H}_{\mathfrak{m}}{}^1(M) \to 0.$$

Analogously, we have isomorphisms for $i \geqslant 1$:

$$\underline{\operatorname{Ext}}_R{}^i(\mathfrak{m}, M) \simeq \underline{\operatorname{Ext}}_R^{i+1}(R/\mathfrak{m}, M)$$

and an exact sequence

$$0 \to \underline{\mathrm{Hom}}_R(R/\mathfrak{m}, M) \to M \to \underline{\mathrm{Hom}}_R(\mathfrak{m}, M) \to \underline{\mathrm{Ext}}_R{}^1(R/\mathfrak{m}, M) \to 0.$$

If M is Noetherian, all $\underline{H}_{\mathfrak{m}}{}^i(M)$ are Artinian graded R-modules. Since $\underline{H}_{\mathfrak{m}}{}^i(M)$ and $H^i_{\mathfrak{m}R_{\mathfrak{m}}}(M_{\mathfrak{m}})$ are isomorphic in a canonical way (over $R_{\mathfrak{m}}$), all properties well known from local algebra are valid, for example, for Noetherian graded R-modules M:

$$\underline{H}_{\mathfrak{m}}{}^i(M) = 0 \qquad \text{for } i > \dim M \ (= \text{Krull dimension of } M)$$

and

$$\underline{H}_{\mathfrak{m}}^{\dim M}(M) \neq 0.$$

Because of the definition of $\underline{H}_{\mathfrak{m}}{}^i(M)$ there is a natural homomorphism for all i

$$\varphi^i \colon \underline{\mathrm{Ext}}_R{}^i(R/\mathfrak{m}, M) \to \underline{H}_{\mathfrak{m}}{}^i(M).$$

Now, let R_1 and R_2 be graded K-algebras with the maximal ideals $\mathfrak{m}_1$ resp. $\mathfrak{m}_2$. Let M_1 and M_2 be graded R_1- resp. R_2-modules. We denote by $\sigma_K(M_1, M_2)$, or briefly $\sigma(M_1, M_2)$, the Segré product of M_1 and M_2 over K. $\sigma_K(M_1, M_2)$ is a graded $\sigma_K(R_1, R_2)$-module with

$$[\sigma_K(M_1, M_2)]_n = [M_1]_n \bigotimes_K [M_2]_n .$$

This definition corresponds to Chow's Segré product of order (1, 1). In this connection we want to point out that the general Segré products of order (d, e) can be reduced to the case $d = 1$, $e = 1$ (see [7, p. 1055]).

Finally we recall some definitions from local algebra. Let $(A, \mathfrak{m})$ be a local ring (with maximal ideal $\mathfrak{m}$) and M a finitely generated A-module. M is called a Cohen–Macaulay module resp. a Buchsbaum module, if for every system of parameters $x_1, \ldots, x_d$ ($d = \dim M > 0$) of M and all $i = 1, \ldots, d$:

$$(x_1, \ldots, x_{i-1})\, M \colon x_i/(x_1, \ldots, x_{i-1})\, M = 0$$

resp.

$$\mathfrak{m}[(x_1, \ldots, x_{i-1})\, M \colon x_i/(x_1, \ldots, x_{i-1})\, M] = 0.$$

The theory of (local) Buchsbaum rings developed from an answer in [21] to a conjecture of Buchsbaum [1, p. 228]. In [18] we found a geometrical interpretation of Buchsbaum rings (see also [19, 14]). In particular, we know that the difference between the length and the multiplicity of any ideal $\mathfrak{q}$ generated by a system of parameters in a (local) Buchsbaum ring is independent of $\mathfrak{q}$, i.e., the difference can be determined by an invariant of A.

Now, let R be a graded K-algebra with the maximal ideal $\mathfrak{m}$ and M a Noetherian graded R-module. M is said to be Cohen–Macaulay module resp. Buchsbaum module if $M_{\mathfrak{m}}$ is Cohen–Macaulay resp. Buchsbaum module over $R_{\mathfrak{m}}$. We know that M is Cohen–Macaulay if and only if $\underline{H}_{\mathfrak{m}}{}^i(M) = 0$ for all $i \neq \dim M$. The main theorem in [20] says that M is a Buchsbaum module if the natural maps φ^i mentioned above are surjective for all $i \neq \dim M$. M is said to be local Cohen–Macaulay resp. local Buchsbaum module, if $M_{\mathfrak{p}}$ is Cohen–Macaulay resp. Buchsbaum module over $R_{\mathfrak{p}}$ for all (homogenous) primes $\mathfrak{p} \neq \mathfrak{m}$ of R.

By depth M resp. dim M we denote the integers depth $M_{\mathfrak{m}}$ resp. dim $M_{\mathfrak{m}}$.

2. The Main Theorem

Theorem. *Let R_1 and R_2 be graded K-algebras and M_1, M_2 graded R_1- resp. R_2-Cohen–Macaulay-modules of finite type with Krull dimensions $d_i \geqslant 2$, $i = 1, 2$.*

(i) *$\sigma(M_1, M_2)$ is a Cohen–Macaulay module (over $\sigma(R_1, R_2)$) if and only if*

$$r(M_1) \leqslant a(M_2) \qquad \text{and} \qquad r(M_2) \leqslant a(M_1);$$

(ii) *$\sigma(M_1, M_2)$ is a Buchsbaum module if and only if*

$$r(M_1) \leqslant a(M_2) + 1 \qquad \text{and} \qquad r(M_2) \leqslant a(M_1) + 1.$$

This Theorem immediately yields the following

Corollary 1. *Let $R = K[x_0, \ldots, x_n]$, $S = K[y_0, \ldots, y_m]$, and $\mathfrak{a} = (f_1, \ldots, f_r) \subset R$, $\mathfrak{b} = (g_1, \ldots, g_s) \subset S$ homogeneous ideals of the principal classes r resp. s with $n - r$, $m - s \geqslant 1$. Then*

(i) *$\sigma(R/\mathfrak{a}, S/\mathfrak{b})$ is a Cohen–Macaulay ring if and only if*

$$\sum_{i=1}^{r} \deg(f_i) \leqslant n \qquad \text{and} \qquad \sum_{j=1}^{s} \deg(g_j) \leqslant m;$$

(ii) *$\sigma(R/\mathfrak{a}, S/\mathfrak{b})$ is a Buchsbaum ring if and only if*

$$\sum_{i=1}^{r} \deg(f_i) \leqslant n + 1 \qquad \text{and} \qquad \sum_{j=1}^{s} \deg(g_j) \leqslant m + 1.$$

The proof is an immediate consequence of the Theorem and the fact that $r(R/\mathfrak{a}) = -n + \sum_{i=1}^{r} \deg(f_i)$, analogous for $r(S/\mathfrak{b})$ (see, for instance, [4, 142]).

Before embarking on the proof of the Theorem and on the constructions of the example in Section 3, we must prove several preliminary results. For these in turn we need Künneth relations for Segré products which seem on the verge of being "well known," but which we cannot locate in the literature. Therefore there is a proof of our Künneth relation in [17]. We remark that Lemma 1 also yields [22, Theorem 4.1.5].

The following lemma yields something more:

LEMMA 1. *Let R_1, R_2 be graded K-algebras with maximal ideals $\mathfrak{m}_1$ and $\mathfrak{m}_2$. Let M_1 and M_2 be graded R_1- resp. R_2-modules. We put $R := \sigma_K(M_1, M_2)$, and*

$$\mathfrak{m}^n := \sigma_K(\mathfrak{m}_1{}^n, \mathfrak{m}_2{}^n) = (\sigma_K(\mathfrak{m}_1, \mathfrak{m}_2))^n.$$

Then for all $i \geqslant 0$ and all $n \geqslant 1$ there are natural homomorphisms

$$\lambda_n{}^i\colon \bigoplus_{r+s=i} \sigma_K(\underline{\mathrm{Ext}}_{R_1}^r(\mathfrak{m}_1{}^n, M_1), \underline{\mathrm{Ext}}_{R_2}^s(\mathfrak{m}_2{}^n, M_2)) \to \underline{\mathrm{Ext}}_R{}^i(\mathfrak{m}^n, M)$$

and for all $i \geqslant 0$ natural isomorphisms

$$\lambda^i\colon \bigoplus_{r+s=i} \sigma_K(\underline{H}^r(M_1), \underline{H}^s(M_2)) \xrightarrow{\sim} \underline{H}^i(M) \qquad \text{(Künneth relations)}$$

High points of the proof. (For the details of the proof see [17].) We take injective resolutions $M_1 \to I_1^{\cdot}$ and $M_2 \to I_2^{\cdot}$. Then the Segré product $\sigma_K(I_1^{\cdot}, I_2^{\cdot})$ yields a (not necessarily injective) resolution of $\sigma_K(M_1, M_2) = M$. Thus we have by [2, p. 83], formula (2) natural homomorphisms

$$\psi_n\colon H^{\cdot}(\mathrm{Hom}(\mathfrak{m}^n, \sigma_K(I_1^{\cdot}, I_2^{\cdot}))) \to \mathrm{Ext}^{\cdot}(\mathfrak{m}^n, M).$$

Applying the Künneth relation from [2, Theorem 3.1, p. 113] we get that the complex $H^{\cdot}(\mathrm{Hom}(\mathfrak{m}^n, \sigma_K(I_1^{\cdot}, I_2^{\cdot})))$ is isomorphic to the complex

$$H^{\cdot}(\mathrm{Hom}(\mathfrak{m}_1{}^n, I_1^{\cdot})) \bigotimes_K H^{\cdot}(\mathrm{Hom}(\mathfrak{m}_2{}^n, I_2^{\cdot})).$$

(Note that $\mathrm{Tor}_1{}^K = 0$ and

$$\mathrm{Hom}(\mathfrak{m}^n, \sigma_K(I_1^{\cdot}, I_2^{\cdot})) \cong \mathrm{Hom}(\mathfrak{m}_1{}^n, I_1^{\cdot}) \bigotimes_K \mathrm{Hom}(\mathfrak{m}_2{}^n, I_2^{\cdot})$$

since for arbitrary graded R_1- resp. R_2-modules M_1 and M_2

$$\mathrm{Hom}(\mathfrak{m}^n, \sigma_K(M_1, M_2)) \cong \mathrm{Hom}(\mathfrak{m}_1{}^n, M_1) \bigotimes_K \mathrm{Hom}(\mathfrak{m}_2{}^n, M_2).)$$

Summarizing all this we get natural homomorphisms

$$\psi_n{}^i\colon \bigoplus_{r+s=i} \left(\mathrm{Ext}_{R_1}^r(\mathfrak{m}_1{}^n, M_1) \bigotimes_K \mathrm{Ext}_{R_2}^s(\mathfrak{m}_2{}^n, M_2)\right) \to \mathrm{Ext}_R{}^i(\mathfrak{m}^n, M).$$

Taking the limit over all n we obtain natural homomorphisms:

$$\psi^i\colon \bigoplus_{r+s=i} \left(H^r(M_1) \bigotimes_K H^s(M_2)\right) \to H^i(M) \qquad \text{for all } i \geqslant 0.$$

It is not hard to show that $H^i(\sigma_K(I_1, I_2)) = 0$ for injective graded modules I_1, I_2 and $i > 0$. By our above remark ψ^0 is also an isomorphism. Therefore by [2, Proposition 4.4, p. 87] ψ^i is an isomorphism for any $i \geqslant 0$. Taking the direct sum over all gradings, we get the statement of Lemma 1. Q.E.D.

Remark. We have examples which show that the natural homomorphisms $\lambda_n{}^i$ for $i \geqslant 1$ are not isomorphisms.

First of all we give two immediate consequences of the Künneth relation:

COROLLARY 1.1. *Let M_1, M_2 graded Noetherian R_1- resp. R_2-modules. Then*

$$\operatorname{depth} \sigma(M_1, M_2) \geqslant \min\{\operatorname{depth} M_1, \operatorname{depth} M_2\}.$$

COROLLARY 1.2. *Let M_1, M_2 be as in the Theorem,* $\mathfrak{m} := \sigma(\mathfrak{m}_1, \mathfrak{m}_2)$. *Then we have:*

(i) $\underline{H}_{\mathfrak{m}}{}^i(\sigma(M_1, M_2)) = 0$ *for all i with* $i \neq d_1, d_2, \quad d_1 + d_2 - 1$;

(ii) $\underline{H}_{\mathfrak{m}}^{d_1}(\sigma(M_1, M_2)) \cong \sigma(\underline{H}_{\mathfrak{m}_1}^{d_1}(M_1), M_2)$,

$$\underline{H}_{\mathfrak{m}}^{d_2}(\sigma(M_1, M_2)) \cong \sigma(M_1, \underline{H}_{\mathfrak{m}_2}^{d_2}(M_2)) \qquad \textit{if} \qquad d_1 \neq d_2,$$

and

$$\underline{H}_{\mathfrak{m}}{}^d(\sigma(M_1, M_2)) \cong \sigma(\underline{H}_{\mathfrak{m}_1}^d(M_1), M_2) \bigoplus \sigma(M_1, \underline{H}_{\mathfrak{m}_2}^d(M_2))$$
$$\textit{if} \qquad d_1 = d_2 =: d.$$

Proof. Note that $\underline{H}^i(M_j) = 0$ for $i \neq 0$ and $i \neq d_j - 1$ and $\underline{H}^0(M_j) \cong M_j$, $\underline{H}^{d_j-1}(M_j) \cong \underline{H}_{\mathfrak{m}_j}^{d_j}(M_j)$ for $j = 1, 2$. The statement follows from the Künneth relations of Lemma 1. Q.E.D.

LEMMA 2. *Let R be a graded K-algebra and M a Noetherian graded R-module with* $d := \dim M \geqslant 1$.

(i) *If* depth $M \geqslant 1$ *then* $[M]_n \neq 0$ *for all* $n \geqslant a(M)$.

(ii) $\mathfrak{m}[\underline{H}_{\mathfrak{m}}{}^d(M)]_n \neq 0$ *for all* $n < e(\underline{H}_{\mathfrak{m}}{}^d(M))$.

Proof. If $[M]_n = 0$ then $[R]_1 [M]_{n-1} \subseteq [M]_n = 0$. Since $\mathfrak{m}$ is generated by $[R]_1$ this implies

$$[M]_{n-1} \subseteq 0\colon_M [R]_1 = 0\colon_M \mathfrak{m} = 0 \qquad (\text{since depth } M \geqslant 1).$$

By induction we have $[M]_i = 0$ for all $i \leqslant n$, i.e., $n < a(M)$. Since $d \geqslant 1$ we have $\underline{H}_{\mathfrak{m}}{}^d(M) \simeq \underline{H}_{\mathfrak{m}}{}^d(M/\underline{H}_{\mathfrak{m}}{}^0(M))$ and thus we may assume depth $M \geqslant 1$. Let x be a homogenous element of degree 1 with $0:{}_M x = 0$. The exact sequence

$$0 \to M \xrightarrow{x} M(1) \to M/xM(1) \to 0$$

yields an epimorphism ($\dim M/xM(1) = d-1$ implies $\underline{H}_{\mathfrak{m}}{}^d(M/xM(1)) = 0$)

$$\underline{H}_{\mathfrak{m}}{}^d(M) \xrightarrow{x} \underline{H}_{\mathfrak{m}}{}^d(M(1)) = \underline{H}_{\mathfrak{m}}{}^d(M)(1).$$

From this we obtain for all $n \in \mathbb{Z}$ epimorphisms

$$[\underline{H}_{\mathfrak{m}}{}^d(M)]_n \xrightarrow{x} [\underline{H}_{\mathfrak{m}}{}^d(M)]_{n+1}.$$

If $\mathfrak{m}[\underline{H}_{\mathfrak{m}}{}^d(M)]_n = 0$, we get by an easy induction argument:

$$[\underline{H}_{\mathfrak{m}}{}^d(M)]_i = 0 \qquad \text{for all } i > n, \qquad \text{i.e.,} \qquad n \geqslant e(\underline{H}_{\mathfrak{m}}{}^d(M)).$$ Q.E.D.

LEMMA 3. *Let R be a graded K-algebra and M a Noetherian graded Cohen–Macaulay R-module. Then*

$$H(n, M) - h(n, M) = (-1)^d \operatorname{rank}_K([H_{\mathfrak{m}}{}^d(M)]_n) \qquad \textit{for all } n \in \mathbb{Z}.$$

Proof. By [16, Nr. 79] we have for arbitrary Noetherian graded R-modules M:

$$h(n, \tilde{M}) = \sum_{i \geqslant 0} (-1)^i \operatorname{rank}_K(H^i(X, \tilde{M}(n))) \qquad \text{for all } n \in \mathbb{Z},$$

where $X := \operatorname{Proj} R$.

The connection with local cohomology yields

$$H(n, M) - h(n, M) = \sum_{i \geqslant 0} (-1)^i \operatorname{rank}_K([\underline{H}_{\mathfrak{m}}{}^i(M)]_n) \qquad \text{for all } n \in \mathbb{Z}.$$

Since M is a Cohen–Macaulay module the assertion follows. Q.E.D.

LEMMA 4. *Let M_1, M_2 be as in the Theorem. Assume further*

$$r(M_1) \leqslant a(M_2) + 1 \qquad \textit{and} \qquad r(M_2) \leqslant a(M_1) + 1.$$

Then the natural homomorphisms ($M := \sigma(M_1, M_2)$, $R := \sigma(R_1, R_2)$)

$$\varphi^i \colon \underline{\operatorname{Ext}}_R{}^i(R/\mathfrak{m}, M) \to \underline{H}_{\mathfrak{m}}{}^i(M)$$

are surjective for all i with $0 \leqslant i < d_1 + d_2 - 1$.

Proof. By Corollary 1.1 we have depth $M \geqslant 2$. Therefore it is sufficient to prove the surjectivity of the corresponding natural homomorphisms

$$\tilde{\varphi}^i\colon \underline{\mathrm{Ext}}_R{}^i(\mathfrak{m}, M) \to \underline{H}^i(M) \qquad \text{for all } i \quad \text{with} \quad 1 \leqslant i < d_1 + d_2 - 2.$$

Let $\tilde{\varphi}_j{}^k\colon \mathrm{Ext}^k_{R_j}(\mathfrak{m}_j, M_j) \to \underline{H}^k(M_j)$ denote the natural maps for $j = 1, 2$ and all k. Analyzing the proof of Lemma 1, we obtain for each i a commutative diagram:

$$\begin{array}{ccc} \bigoplus_{r+s=i} \sigma(\underline{\mathrm{Ext}}^r_{R_1}(\mathfrak{m}_1, M_1), \underline{\mathrm{Ext}}^s_{R_2}(\mathfrak{m}_2, M_2)) & \xrightarrow{\bar{\varphi}^i} & \bigoplus_{r+s=i} \sigma(\underline{H}^r(M_1), \underline{H}^s(M_2)) \\ \downarrow \lambda_1^i & & \downarrow \wr\, \lambda^i \\ \underline{\mathrm{Ext}}_R{}^i(\mathfrak{m}, M) & \xrightarrow{\tilde{\varphi}^i} & \underline{H}^i(M) \end{array}$$

where

$$\bar{\varphi}^i := \bigoplus_{r+s=i} \sigma(\tilde{\varphi}_1{}^r, \tilde{\varphi}_2{}^s).$$

But $\sigma(\underline{H}^r(M_1), \underline{H}^s(M_2)) = 0$ for $(r, s) \neq (0, 0), (0, d_2 - 1), (d_1 - 1, 0), (d_1 - 1, d_2 - 1)$ (see Corollary 1.2). Note that we may assume $\tilde{\varphi}_j{}^0 = id$ for $j = 1, 2$. We want to show that $\sigma(\tilde{\varphi}_1^{d_1-1}, id)$ and $\sigma(id, \tilde{\varphi}_2^{d_2-1})$ are surjective. Then we are done because this implies the surjectivity of $\bar{\varphi}^i$ and thus by our commutative diagram the surjectivity of $\tilde{\varphi}^i$ for each i with $1 \leqslant i < d_1 + d_2 - 2$.

For this reason let us assume $r(M_1) \leqslant a(M_2) + 1$. We put $a := a(M_2)$, $e := e(\underline{H}^{d_1-1}(M_1)) = e(\underline{H}^{d_1}_{\mathfrak{m}_1}(M_1))$. By Lemma 3 we have $r(M_1) = e + 1$ and consequently $e \leqslant a$ by our assumption. If $e < a$ then $\sigma(\underline{H}^{d_1-1}(M_1), M_2) = 0$ and $\sigma(\tilde{\varphi}_1^{d_1-1}, id)$ is surjective. Assume $e = a$. Then

$$[\sigma(\underline{H}^{d_1-1}(M_1), M_2)]_n = 0 \qquad \text{for } n \neq e$$

and

$$[\sigma(\underline{H}^{d_1-1}(M_1), M_2)]_e = [\underline{H}^{d_1-1}(M_1)]_e \bigotimes_K [M_2]_a\,.$$

$\tilde{\varphi}_1^{d_1-1}$ is the embedding

$$\underline{\mathrm{Ext}}^{d_1-1}_{R_1}(\mathfrak{m}_1, M_1) \cong \underline{\mathrm{Hom}}_{R_1}(R_1/\mathfrak{m}_1, \underline{H}^{d_1-1}(M_1)) \subset \underline{H}^{d_1-1}(M_1).$$

But since $\mathfrak{m}_1[\underline{H}^{d_1-1}(M_1)]_e = 0$, $[\tilde{\varphi}_1^{d_1-1}]_e$ is an isomorphism. Hence $\sigma(\tilde{\varphi}_1^{d_1-1}, id)$ is surjective (even an isomorphism). The same arguments yield the surjectivity of $\sigma(id, \tilde{\varphi}_2^{d_2-1})$. Q.E.D.

Proof of the Theorem. (i) Corollary 1.2 shows that $\sigma(M_1, M_2)$ is a Cohen–Macaulay module if and only if

$$\sigma(\underline{H}^{d_1}_{\mathfrak{m}_1}(M_1), M_2) = \sigma(M_1, \underline{H}^{d_2}_{\mathfrak{m}_2}(M_2)) = 0.$$

Using Lemma 2 we see that this is possible if and only if $e(\underline{H}^{d_1}_{\mathfrak{m}_1}(M_1)) < a(M_2)$

and $e(\underline{H}^{d_2}_{\mathfrak{m}_2}(M_2)) < a(M_1)$. But with Lemma 3 we have for $i = 1, 2$: $e(\underline{H}^{d_i}_{\mathfrak{m}_i}(M_i)) = r(M_i) - 1$. This proves statement (i).

(ii) By Lemma 4 φ^i is surjective for all i with $0 \leqslant i < d_1 + d_2 - 1$. From [20, Theorem 1] we get that $\sigma(M_1, M_2)$ is a Buchsbaum module.

Assume now that $\sigma(M_1, M_2)$ is a Buchsbaum module. Then [14, Lemma 3] yields $\mathfrak{m} \cdot \underline{H}_{\mathfrak{m}}{}^i(\sigma(M_1, M_2)) = 0$ for all $i = 0, \dots, d_1 + d_2 - 2$. Using Corollary 1.2 we get

$$0 = \mathfrak{m} \cdot \sigma(\underline{H}^{d_1}_{\mathfrak{m}_1}(M_1), M_2) = \sigma(\mathfrak{m}_1 \underline{H}^{d_1}_{\mathfrak{m}_1}(M_1), \mathfrak{m}_2 M_2).$$

Therefore Lemma 2 yiels $e(\underline{H}^{d_1}(M_1)) \leqslant a(M_2)$ and by Lemma 3 we obtain $r(M_1) - 1 \leqslant a(M_2)$. Analogously we get $r(M_2) - 1 \leqslant a(M_1)$. This proves statement (ii). Q.E.D.

Proof of the Corollary stated in the Introduction. Let $R_1 = M_1$ and $R_2 = M_2$ be the homogeneous coordinate ring of V and W, respectively. We set $\dim(V) = d$ and $\dim(W) = \delta$. Since $a(R_i) = 0$ we obtain by the application of the Theorem that $S(V \times W)$ is an arithmetically Buchsbaum variety if and only if $r(R_1) =: r(V) \leqslant 1$ and $r(R_2) =: r(W) \leqslant 1$.

$S(V \times W)$ is an arithmetically Cohen–Macaulay variety if and only if $r(V) \leqslant 0$ and $r(W) \leqslant 0$. Therefore we have:

$$1 = H(0, R_1) = h(0, R_1) = h_d \qquad \text{and} \qquad 1 = H(0, R_2) = h(0, R_2) = h_\delta,$$

where h_d and h_δ are the coefficients of the binomial coefficient $\binom{t}{0}$ of Hilbert's, characteristic polynomial $h(t, R_1)$ and $h(t, R_2)$, respectively. Since the arithmetic genus of V, $p_a(V)$ say, is given by $p_a(V) = (-1)^d (h_d - 1)$ (see, e.g., [5] or [23]) we get $p_a(V) = p_a(W) = 0$. If $p_a(V) = p_a(W) = 0$ we again obtain: $H(0, R_1) = 1 = h_d = h(0, R_1)$ and $H(0, R_2) = 1 = h_\delta = h(0, R_2)$. Therefore Lemma 3 yields that $[\underline{H}^{d+1}_{\mathfrak{m}_1}(R_1)]_0 = [\underline{H}^{\delta+1}_{\mathfrak{m}_2}(R_2)]_0 = 0$. Statement (ii) of Lemma 2 shows that $e(\underline{H}^{d+1}_{\mathfrak{m}_1}(R_1)), e(\underline{H}^{d+1}_{\mathfrak{m}_2}(R_2)) < 0$. Applying Lemma 3 we get that $H(n, R_i) = h(n, R_i)$ for all $n \geqslant 0$ and $i = 1, 2$, i.e., $r(V), r(W) \leqslant 0$. Q.E.D.

We now want to relate our results to some work of Chow [3]. We still need the following:

LEMMA 5. *Let R be a graded Cohen–Macaulay K-algebra with dimension m and maximal ideal $\mathfrak{m}$. Then we have:*

$$m + r(R) = \inf_t \{t \mid \mathfrak{m}^t \subseteq (x_1, \dots, x_m), \textit{ where } x_1, \dots, x_m \textit{ is an arbitrary system of parameters consisting of forms of degree } 1\} =: t_0.$$

Proof. We set $r =: r(R)$ and $S =: R/(x_1, \dots, x_m)$. We first show

(1) $t_0 \leqslant m + r$: Since R is Cohen–Macaulay we have:

$$\sum_{i=0}^{r} \binom{r-i+m+1}{m-1} H(i, S) = H(r, R) = h(r, R)$$

$$= \sum_{i\geqslant 0} \binom{r-i+m-1}{m-1} H(i, S).$$

Of course, we have $\binom{r-i+m-1}{m-1} = 0$ for $r + 1 \leqslant i \leqslant r + m - 1$, and for $i \geqslant r + m$:

$$\binom{r-i+m-1}{m-1} \begin{cases} < 0 & \text{if} \quad m \text{ is even}, \\ > 0 & \text{if} \quad m \text{ is odd}. \end{cases}$$

From this it follows that $H(i, S) = 0$ for all $i \geqslant r + m$, i.e., $\mathfrak{m}^{r+m} \subseteq (x_1, ..., x_m)$. We now show

(2) $t_0 \geqslant m + r$: Since $\mathfrak{m}^{t_0} \subseteq (x_1, ..., x_m)$ we have $H(i, S) = 0$ for $i \geqslant t_0$. From this it follows that $h(t, R) = \sum_{i=0}^{t_0-1} \binom{t-i+m-1}{m-1} H(i, S)$, i.e., $h(t, R) = H(t, R)$ if $t \geqslant t_0 - 1$. If $t_0 - m \leqslant t < t_0 - 1$ then we have for all i with $t < i \leqslant t_0 - 1$: $\binom{t-i+m-1}{m-1} = 0$, i.e., $h(t, R) = \sum_{i=0}^{t} \binom{t-i+m-1}{m-1} H(i, S) = H(t, R)$ for all $t \geqslant t_0 - m$, and we obtain $r(R) \leqslant t_0 - m$. Q.E.D.

Remark. Using this Lemma we can calculate the regularity index of a perfect homogeneous polynomial ideal without knowing Hilbert's characteristic function (see Example 3).

We are now in a position to prove the main theorem of [3] for our graded K-algebras.

COROLLARY 2. *Let R_1 and R_2 be graded Cohen–Macaulay K-algebras with* $\dim(R_1) =: d_1 \geqslant 1$, $\dim(R_2) =: d_2 \geqslant 1$. *The Segré product $\sigma(R_1, R_2)$ is a Cohen–Macaulay K-algebra if and only if R_1 and R_2 are proper, i.e., there exist systems of parameters $x_1, ..., x_{d_1} \in R_1$ and $y_1, ..., y_{d_2} \in R_2$ consisting of forms of degree 1 such that $\mathfrak{m}_1^{d_1} \subseteq (x_1, ..., x_{d_1})$ and $\mathfrak{m}_2^{d_2} \subseteq (y_1, ..., y_{d_2})$.*

Proof. Lemma 5 shows that $d_i + r(R_i) \leqslant d_i$ for $i = 1, 2$. Therefore statement (i) of the Theorem yields the Corollary. Q.E.D.

These previous results show that the arithmetically Cohen–Macaulay property (and the Buchsbaum property) is not transferred in general to the Segré product. Therefore we look into the local behavior in respect to the Cohen–Macaulay property. One obtains essentially from the Künneth relation the following result for a geometrical application.

PROPOSITION 6. *Let $V \subset \mathbb{P}^n$ and $W \subset \mathbb{P}^m$ be varieties with dimension $d \geqslant 1$*

and $\delta \geqslant 1$, respectively. The Segré embedding $S(V \times W)$ is a locally Cohen–Macaulay variety if and only if V and W are locally Cohen–Macaulay varieties.

Proof. Let R_1, R_2, and R be the homogeneous coordinate rings of V, W, and $S(V \times W)$, respectively. We have $\underline{H}^i(R_1) = \oplus_{p\in\mathbb{Z}} \underline{H}^i(V, O_V(p))$ and analogous $\underline{H}^i(R_2)$ and $\underline{H}^i(R)$. Using the Künneth relation from Lemma 1 we get

$$\underline{H}^i(R) = \bigoplus_{r+s=i} \sigma(\underline{H}^r(R_1), \underline{H}^s(R_2)) \qquad \text{for all } i \geqslant 0.$$

(1) Suppose now that V and W are locally Cohen–Macaulay. Then we know that the modules $\underline{H}^r(R_1)$ and $H^s(R_2)$ have finite length over K for $r = 1,..., d-1$ and $s = 1,..., \delta - 1$ (see, e.g., [8] or [20]). However, we have the following exact sequence for $i = 1, 2$:

$$0 \to \underline{H}^0_{\mathfrak{m}_i}(R_i) \to R_i \to \underline{H}^0(R_i) \to \underline{H}^1_{\mathfrak{m}_i}(R_i) \to 0, \qquad (*)$$

where also $\underline{H}^0_{\mathfrak{m}_i}(R_i)$ and $\underline{H}^1_{\mathfrak{m}_i}(R_i)$ have finite length, which shows that $[\underline{H}^0(R_i)]_n = 0$ for $n \ll 0$. If $1 \leqslant r+s=i \leqslant d+\delta-1$ then we see that $[\sigma(\underline{H}^r(R_1), \underline{H}^s(R_2))]_n \neq 0$ only for finite many n, i.e., $\underline{H}^i(R)$ has finite length for $1 \leqslant i \leqslant d+\delta-1$. Since $[\underline{H}^0(R)]_n = [\sigma(\underline{H}^0(R_1), \underline{H}^0(R_2)]_n = [\underline{H}^0(R_1)]_n \otimes_K [\underline{H}^0(R_2)]_n$ we obtain that $[\underline{H}^0(R)]_n = 0$ for $n \ll 0$. Therefore it follows from the exact sequence $0 \to \underline{H}_{\mathfrak{m}}{}^0(R) \to R \to \underline{H}^0(R) \to \underline{H}_{\mathfrak{m}}{}^1(R) \to 0$ that $\underline{H}_{\mathfrak{m}}{}^0(R)$ and $\underline{H}_{\mathfrak{m}}{}^1(R)$ have finite length. Hence we see that $\underline{H}^i(R)$ has finite length for $i = 0, 1,..., d+\delta$, i.e., the Segré embedding $S(V \times W)$ is a locally Cohen–Macaulay variety (see, for example, [8]).

(2) Without loss of generality we assume that V is not locally Cohen–Macaulay. Then there exists an integer r with $0 \leqslant r \leqslant d-1$ such that $\underline{H}^{r+1}_{\mathfrak{m}_1}(R_1)$ has not finite length, i.e., we have $[\underline{H}^{r+1}_{\mathfrak{m}_1}(R_1)]_{-n} \neq 0$ for infinite many positive integers n. This is also true for $[\underline{H}^r(R_1)]$ because $\underline{H}^{r+1}_{\mathfrak{m}_1}(R_1) \simeq \underline{H}^r(R_1)$ for each $r \geqslant 1$ and we have the exact sequence (*) for $r = 0$. Now the Segré product $\sigma(\underline{H}^r(R_1), \underline{H}^\delta(R_2))$ has the direct summand $H^{r+\delta}(R)$. Since $1 \leqslant r+\delta < d+\delta$ it is sufficient to show that this summand has not finite length. This statement but follows directly from Lemma 2(ii) applied to $\underline{H}^{\delta+1}_{\mathfrak{m}_2}(R_2) \simeq \underline{H}^\delta(R_2)$.

From this it follows that $\underline{H}^{r+\delta+1}_{\mathfrak{m}}(R)$ has not finite length and this is the required contradiction. Q.E.D.

3. Examples

PROPOSITION 7. *Let $V \subset \mathbb{P}_K{}^n$ and $W \subset \mathbb{P}_K{}^n$ be locally Buchsbaum varieties. It is not true in general that the Segré embedding $S(V \times W)$ is a locally Buchsbaum variety.*

EXAMPLE 1. Let $V \subset \mathbb{P}^4$ be the surface given parametrically by

$$\{t_0^3, t_0^2 t_1, t_0 t_1 t_2, t_0 t_2(t_2 - t_0), t_0^2(t_2 - t_0)\}.$$

We know that V is a locally Buchsbaum variety, by [14, Corollary 7]. Let X be the Segré embedding $S(V \times \mathbb{P}^1) \subset \mathbb{P}^9$.

Claim. The local ring of X at the origin of $\mathbb{P}^9$ is a non-Buchsbaum ring, i.e., X is not a locally Buchsbaum variety.

This statement proves our Proposition 7.

Proof. The surface V is given by the imperfect ideal

$$\begin{aligned}\mathfrak{p}_V =: (&x_1x_4 - x_2x_3, x_0x_1x_2 - x_0x_2^2 + x_1^2x_3, x_0x_2x_3 - x_0x_2x_4 + x_1x_3^2,\\ &x_0x_3x_4 - x_0x_4^2 + x_3^3) \subset K[x_0, \dots, x_4]\end{aligned}$$

(see [12, p. 334]).

Therefore the Segré embedding X is given by the following ideal:

$$\begin{aligned}\mathfrak{p}_X =: (&x_2x_8 - x_4x_6, x_2x_9 - x_4x_7, x_3x_9 - x_5x_7, x_0x_2x_4 - x_0x_4^2 + x_2^2x_6,\\ &x_0x_2x_5 - x_1x_4^2 + x_2^2x_7, x_0x_3x_5 - x_0x_5^2 + x_6x_3^2, x_1x_3x_5 - x_1x_5^2 + x_3^2x_7,\\ &x_0x_4x_6 - x_0x_4x_8 + x_2x_6^2, x_0x_4x_7 - x_0x_4x_9 + x_3x_6^2, x_0x_5x_7 - x_0x_5x_9 + x_2x_7^2,\\ &x_1x_5x_7 - x_1x_5x_9 + x_3x_7^2, x_0x_6x_8 - x_0x_8^2 + x_6^3, x_0x_6x_9 - x_1x_8^2 + x_6^2x_7,\\ &x_0x_7x_9 - x_0x_9^2 + x_6x_7^2, x_1x_7x_9 - x_1x_9^2 + x_7^3, x_0x_3 - x_1x_2, x_0x_5 - x_1x_4,\\ &x_0x_7 - x_1x_6, x_0x_9 - x_1x_8, x_3x_4 - x_2x_5, x_4x_7 - x_5x_6, x_4x_9 - x_5x_8,\\ &x_3x_6 - x_2x_7, x_6x_9 - x_7x_8, x_3x_8 - x_2x_9) \subset K[x_0, x_1, \dots, x_9].\end{aligned}$$

Let A be the local ring of X at the origin $x_1 = \cdots = x_9 = 0$, and let B be the local ring of X at the subvariety $x_2 = \cdots = x_9 = 0$. We will show that B is a non-Cohen–Macaulay ring. From this we get that A is a non-Buchsbaum ring (see [18] or [19]). We consider the radical of $(\mathfrak{p}_X, x_2)$ in B, $\operatorname{rad}((\mathfrak{p}_X, x_2) \cdot B)$. Let Y be a variety of $\mathbb{P}_K^2$ defined by $z_0z_1z_2 - z_0z_2^2 + z_1^3 = 0$. Let W be the Segré embedding $S(Y \times \mathbb{P}^1) \subset \mathbb{P}^5$ defined by the ideal, $\mathfrak{p}_W$ say. Then we obtain by a suitable numeration of the indeterminates that $\operatorname{rad}((\mathfrak{p}_X, x_2) \cdot B) = (x_2, x_3, x_4, x_5, \mathfrak{p}_W) \cdot B$, i.e., a primary decomposition of $(\mathfrak{p}_X, x_2) \cdot B$ has only one isolated component. This component is given by $\operatorname{rad}((\mathfrak{p}_X, x_2) \cdot B)$, because $x_2, x_3, x_4x_6, x_6x_5, \mathfrak{p}_W \in (\mathfrak{p}_X, x_2) \cdot B$, and therefore the ideal $(\mathfrak{p}_X, x_2) \cdot B$ has an embedded component. This shows that B is a non-Cohen–Macaulay ring. In the fact, one can show that $l(B/\mathfrak{q}) = 3$ and $e_0(\mathfrak{q}, B) = 2$, where $\mathfrak{q} \subset B$ is generated by the system of parameters x_2, x_9. Q.E.D.

Remark. Studying a primary decomposition of $(\mathfrak{p}_X, x_2^2) \cdot B$ we get from Theorem 5(iii) in [19] that B is a Buchsbaum ring.

PROPOSITION 8 (see Seidenberg [15]). *Let G, H be plane curves without singularities, at least one of which has positive genus. The Segré embedding of $G \times H$ is an arithmetically normal non-Cohen–Macaulay surface F free of singularities.*

Proof. The arithmetic genus of G, H is the usual genus of G, H. The Theorem therefore yields that F is an arithmetically non-Cohen–Macaulay surface. Let A be the local ring of the vertex of the affine cone over F. The Corollary 1.1 yields that $\operatorname{depth}(A) \geqslant 2$. Of course, F is free of singularities. Therefore Serre's characterization of normal rings (see, e.g., [11, p. 125]) shows that F is an arithmetically normal surface. Q.E.D.

From this we obtain immediately an example from [20].

EXAMPLE 2. Let G be the cubic curve defined by the equation $x_1^2x_2 - x_0^3 + x_0x_2^2 = 0$ in $\mathbb{P}^2$ and let $H = \mathbb{P}^1$. Let F be $G \times H$ in its Segré embedding in $\mathbb{P}^5$. Let A be the local ring of the vertex of the affine cone over F. By Proposition 8, A is a normal non-Cohen–Macaulay ring. By Corollary 1 of the Theorem we obtain that A is a Buchsbaum ring.

But our Theorem also permits us to work with incomplete intersections in order to construct normal Buchsbaum rings which are not Cohen–Macaulay.

EXAMPLE 3. Let $C \subset \mathbb{P}_K{}^3$ be the curve given parametrically by

$$\{t_0^5, t_0^4t_1 + t_1^5, t_0^3t_1^2, t_0t_1^4\}.$$

Claim. Let A be the local ring of the vertex of the affine cone over the Segré embedding of $C \times \mathbb{P}^1$. Then A is a normal Buchsbaum ring but not a Cohen–Macaulay ring.

Proof. The curve C is arithmetically Cohen–Macaulay (see, e.g., [13]). Therefore Corollary 1.1 yields that $\operatorname{depth}(A) \geqslant 2$. It is not hard to show that C is nonsingular. Hence A is a normal ring (by Serre's criterion). Let R be the homogeneous coordinate ring of C. We will show that $r(C) = 1$.

The curve C is defined by the prime ideal

$$\mathfrak{p} =: (x_0x_3 - x_2^2, x_0^2x_2 - x_0x_1^2 + 2x_2^3 + x_2x_3^2, x_0x_2^2 - x_1^2x_2 + 2x_2^2x_3 + x_3^3).$$

Hence the elements x_2, $x_0 + x_1$ are a system of parameters of R of degree 1, and we can show that

$$(x_0, x_1, x_2, x_3)^3 \subseteq (x_1, x_0 + x_1) + \mathfrak{p}$$

but

$$(x_0, x_1, x_2, x_3)^2 \nsubseteq (x_2, x_0 + x_1) + \mathfrak{p}$$

because $x_3{}^2 \notin (x_2, x_0 + x_1) + \mathfrak{p}$. Using Lemma 5 we get that $r(R) = 1$. The Theorem yields that A is a Buchsbaum but non-Cohen–Macaulay ring. Q.E.D.

PROPOSITION 9. *Let d be an arbitrary integer with $d \geqslant 3$. Then there exist projective varieties $V \subset \mathbb{P}^K_{2d-1}$ such that the local ring A of the vertex of the affine cone over V is a normal Buchsbaum ring but not a Cohen–Macaulay ring of dimension d.*

EXAMPLE 4. Let $W \subset \mathbb{P}^{d-1}$ be the variety defined by the equation $x_0{}^d + x_1{}^d + \cdots + x_{d-1}^d = 0$. Let A be the local ring of the vertex of the affine cone over the Segré embedding of $W \times \mathbb{P}^1 \subset \mathbb{P}^{2d-1}$. Then we obtain that $\dim(A) = d$ and $\operatorname{depth}(A) \geqslant 2$, by Corollary 1.1. Since W is nonsingular we get that A is a normal ring (by Serre's criterion). Corollary 1 yields that A is a Buchsbaum non-Cohen–Macaulay ring.

Remarks. (1) With considerable effort the proof is given that A is a normal non-Cohen–Macaulay ring for the case $d = 3$ in [9, Theorem 1] and [10, Theorem 2]. In [10] Markoe even applies analytical methods; for example, he uses a generalization of Oka's hypersurface normality criterion. An easy generalization of this case is also given in [6] (see Example 5.9).

(2) In view of Proposition 9 we mention the following remarks of Gunning and Markoe (see [10]): "Examples of normal germs having relatively large homological dimension are apparently rather harder to come by ⋯ Examples of non-perfect normal varieties are hard to construct and hard to find in the literature." In this connection we still want to point out that Hironaka asked the following question in a discussion (at the University of Halle): Do there exist normal Buchsbaum singularities which are not Cohen–Macaulay (see [20])?

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